Time-frequency representations

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Outline

Introduction

Classical time-frequency representations
  Short-time Fourier transforms
  Wigner distribution
  Altes Q distribution
  Cross-term problems

Classes of TFRs
  Classes of TFRs with common properties
  Shift-covariant class
  Affine class
  Hyperbolic and kth power classes

Discrete calculations of TFRs
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Discrete calculations of TFRs
First... Fourier: the Fourier transform (again!)
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First... Fourier: the Fourier transform (again!)

\[ X(f) = \int x(t)e^{-j2\pi ft} dt \]
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\[ x(t) \longleftrightarrow X(f) \]
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The main lack of the Fourier analysis

Since sinusoidal basis functions are infinite duration waveforms...
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Fourier analysis implicitly assumes that each sinusoidal component with nonzero weighting coefficient is always present...
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Fourier analysis implicitly assumes that each sinusoidal component with nonzero weighting coefficient is always present

This is, that the spectral content of the signal under analysis is *stationary*...
Time, frequency, Fourier and time-frequency
Instantaneous frequency

Instantaneous Frequency is a time-varying frequency analysis...
**Instantaneous frequency**

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The instantaneous frequency of a time-varying signal is defined as the instantaneous change in the *phase* of that signal

\[ f_x(t) = \frac{1}{2\pi} \frac{d}{dt} \text{arg} \, x(t) \]
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Graphical interpretation: phase shift...
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Graphical interpretation: phase shift...

For example, if \( x(t) = e^{j2\pi(\alpha t)} \), its instantaneous frequency is

\[ f_x(t) = 2\alpha t \]
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Instantaneous frequency

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\[ y(t) = e^{j2\pi f_1 t} + e^{j2\pi f_2 t} \]
**Instantaneous frequency**

However, this time-varying spectral representation may be counterintuitive for multicomponent signals. For example, from

$$y(t) = e^{j2\pi f_1 t} + e^{j2\pi f_2 t}$$

can be derived that

$$f_y(t) = \frac{f_1 + f_2}{2}$$

(an interesting property...)
Instantaneous frequency

Now, what is the instantaneous frequency of

\[ y(t) = e^{j2\pi f_1 t} + e^{j2\pi (-f_1)t} \ldots? \]
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\[ f_y(t) = \frac{f_1 - f_1}{2} = 0 \]
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But note that

\[ y(t) = \cos(2\pi f_1 t) + \cos(2\pi (-f_1)t) \]
Instantaneous frequency

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But note that

$$y(t) = \cos(2\pi f_1 t) + \cos(2\pi (-f_1) t)$$
$$= \cos(2\pi f_1 t) + \cos(2\pi f_1 t)$$
$$= 2 \cos(2\pi f_1 t) \ldots$$
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But note that

\[ y(t) = \cos(2\pi f_1 t) + \cos(2\pi(-f_1)t) \]
\[ = \cos(2\pi f_1 t) + \cos(2\pi f_1 t) \]
\[ = 2 \cos(2\pi f_1 t) \ldots \]

Then the real signal \(2 \cos(2\pi f_1 t)\) has an instantaneous frequency equal to zero...?
General motivations

Nonstationary signals $\implies$ time-varying spectral content.
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¿How the spectral content of a signal is changing with time?
General motivations

Nonstationary signals $\Rightarrow$ time-varying spectral content.

¿How the spectral content of a signal is changing with time?

TFRs: from a one-dimensional signal $x(t)$
$\rightarrow$ we look for a two-dimensional function $T_x(t, f)$
Nonstationary signals \( \implies \) time-varying spectral content.

¿How the spectral content of a signal is changing with time?

TFRs: from a one-dimensional signal \( x(t) \)
\[ \implies \] we look for a two-dimensional function \( T_x(t, f) \)

The time-frequency plane is like a *musical score*...
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Discrete calculations of TFRs
Short-time Fourier transforms (STFT)

\[ S_x(t, f; \Gamma) = \int x(\tau) \gamma^*(\tau - t) e^{-j2\pi f \tau} d\tau \]
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Remarks...

- \( S_x(t, f; \Gamma) \) is a linear transform of \( x(t) \)
Short-time Fourier transforms (STFT)

\[ S_x(t, f; \Gamma) = \int x(\tau) \gamma^*(\tau - t) e^{-j2\pi f \tau} d\tau \]

Remarks...

- \( S_x(t, f; \Gamma) \) is a linear transform of \( x(t) \)
- Typically, the analysis window, \( \gamma(t) \), is real and even
Short-time Fourier transforms (STFT)

\[ S_x(t, f; \Gamma) = \int x(\tau) \gamma^*(\tau - t) e^{-j2\pi f \tau} d\tau \]

\[ = e^{-j2\pi tf} \int X(\zeta) \Gamma^*(\zeta - f) e^{j2\pi t\zeta} d\zeta \]

Remarks...

- \( S_x(t, f; \Gamma) \) is a **linear** transform of \( x(t) \)
- Typically, the analysis window, \( \gamma(t) \), is real and even
- STFT can also be thought of as the temporal fluctuations of the signal spectrum near the output frequency \( f \).
STFT: the spectrogram

The STFT used for speech in the old *analog* analyzers was originally known as “the spectrogram”
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\[
G_x(t, f; \Gamma) = |S_x(t, f; \Gamma)|^2 = \left| \int x(\tau) \gamma^*(\tau - t) e^{-j2\pi f \tau} d\tau \right|^2 = \left| \int X(\zeta) \Gamma^*(\zeta - f) e^{j2\pi t \zeta} d\zeta \right|^2
\]

...still linear?
STFT: example 1

Consider the case of $x(t) = \delta(t - t_0)$
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\[
S_x(t, f; \Gamma) = \int \delta(\tau - t_0) \gamma^*(\tau - t) e^{-j2\pi f\tau} d\tau
\]

\[
= \gamma^*(t_0 - t) e^{-j2\pi ft_0}
\]
Consider the case of $x(t) = \delta(t - t_0)$

$$S_x(t, f; \Gamma) = \int \delta(\tau - t_0)\gamma^*(\tau - t)e^{-j2\pi ft}\,d\tau$$

$$= \gamma^*(t_0 - t)e^{-j2\pi ft_0}$$

Remarks...

- The window is still in time (not the transform of the window)
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- The window is the same for all frequencies.
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- The window is centered at \( t_0 \) and its support has not changed.
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- Graphical interpretation...
STFT: example 2

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$$S_y(t, f; \Gamma) = e^{-j2\pi tf} \int \delta(\zeta - f_0) \Gamma^*(\zeta - f) e^{j2\pi t\zeta} d\zeta$$

$$= \Gamma^*(f_0 - f) e^{-j2\pi(f - f_0)t}$$
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Remarks...

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- Graphical interpretation...

*Homework:* Uncertainty principle [1] 7.1.3

*Homework:* Reconstruction [1] 7.1.5
**STFT: main advantage**

The STFT is a linear transformation of the signal

\[
y(t) = \alpha x_1(t) + \beta x_2(t) \]

\[
\uparrow
\]

\[
S_y(t, f; \Gamma) = \alpha S_{x_1}(t, f; \Gamma) + \beta S_{x_2}(t, f; \Gamma)
\]
STFT: drawbacks

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- Long duration and wide bandwidth: producing a spreading of the time-frequency support
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- STFT is complex-valued

- Long duration and wide bandwidth: producing a spreading of the time-frequency support

- Tradeoff between time resolution and frequency resolution: achieving either good time resolution or good frequency resolution but generally not both
STFT: example 3

Consider the two component signal

\[ q(t) = \delta(t - t_0) + e^{j2\pi f_0 t} \]
STFT: example 3

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\[ q(t) = \delta(t - t_0) + e^{j2\pi f_0 t} \]

Using a Gaussian window

\[ \gamma(t) = \frac{1}{\sqrt{\sigma}} e^{-\pi(t/\sigma)^2} \]

\[ \Gamma(f) = \sqrt{\sigma} e^{-\pi(\sigma f)^2} \]

(i.e., the STFT becomes the Gabor transform)
It can be show that

\[ S_q(t, f; \Gamma) = \frac{1}{\sqrt{\sigma}} e^{-\pi(t-t_0)^2/\sigma^2} e^{-j2\pi ft_0} + \sqrt{\sigma} e^{-\pi\sigma^2(f-f_0)^2} e^{-j2\pi(f-f_0)t} \]
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Remarks...

- Ideally, it should be highly concentrated around \( t_0 \) and \( f_0 \), but...
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- In order to be concentrated in \( t_0 \), we need \( \sigma \to 0 \)
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Remarks...

- Ideally, it should be highly concentrated around \( t_0 \) and \( f_0 \), but...
- In order to be concentrated in \( t_0 \), we need \( \sigma \to 0 \)
- In order to be concentrated in \( f_0 \), we need \( \sigma \gg 0 \)
Wigner Distribution (WD)

\[ W_x(t, f) = \int x \left( t + \frac{T}{2} \right) x^* \left( t - \frac{T}{2} \right) e^{-j2\pi f \tau} d\tau \]

\[ = \int X \left( f + \frac{\zeta}{2} \right) X^* \left( f - \frac{\zeta}{2} \right) e^{j2\pi t \zeta} d\zeta \]
WD: the support of $x(s + t/2)x(s - t/2)$

What is the support if the WD don’t have a window?
**WD: the support of** \( x(s + t/2)x(s - t/2) \)

Imagining that a signal’s support lies mainly within the interval \([-a, a]\)...
WD: the support of $x(s + t/2)x(s - t/2)$
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WD: properties

\[ x(t) \quad W_x(f, t) \]

\[ \delta(t - t_i) \quad \delta(t - t_i) \]

\[ e^{j2\pi f_i t} \quad \delta(f - f_i) \]

\[ e^{+j\pi \alpha t^2} \quad \delta(f - \alpha t) \]

\[ \frac{1}{\sqrt{j\alpha}} e^{j\pi i^2/\alpha} \quad \delta(t - \alpha f') \]

\[ e^{j\pi (\alpha t^2 + 2f_it + c)} \quad \delta(f - f_i - \alpha t) \]

\[ \frac{1}{\sqrt{\sigma}} e^{-\pi (t/\sigma)^2} \quad \sqrt{2}e^{-2\pi [(t/\sigma)^2 + (\sigma f)^2]} \]

\[ \frac{1}{\sqrt{\sigma}} e^{-\pi (t/\sigma)^2} e^{j\pi \alpha t^2} \quad \sqrt{2}e^{-2\pi [(t/\sigma)^2 + (\sigma f)^2] + \sigma^2(f - f_i)^2} \]

\[ \frac{1}{\sqrt{\sigma}} e^{-\pi [(t - t_i)/\sigma]^2} e^{j2\pi f_i t} \quad \sqrt{2}e^{-2\pi [[(t - t_i)/\sigma]^2 + \sigma^2(f - f_i)^2]} \]

\[ \text{rect}_a(t) \quad \frac{\sin[4\pi(a - |t|)f]}{\pi f} \text{rect}_a(t) \]
**WD: properties**

\[
\begin{align*}
  x(t) & \quad W_x(f, t) \\
  \delta(t - t_i) & \quad \delta(t - t_i) \\
  e^{j2\pi f_i t} & \quad \delta(f - f_i) \\
  e^{j\pi \alpha t^2} & \quad \delta(f - \alpha t) \\
  \frac{1}{\sqrt{j\alpha}} e^{j\pi i^2/\alpha} & \quad \delta(t - \alpha f) \\
  e^{j\pi (\alpha t^2 + 2 f_i t + c)} & \quad \delta(f - f_i - \alpha t) \\
  \frac{1}{\sqrt{\sigma}} e^{-\pi(t/\sigma)^2} & \quad \sqrt{2} e^{-2\pi[(t/\sigma)^2 + (\sigma f)^2]} \\
  \frac{1}{\sqrt{\sigma}} e^{-\pi(t/\sigma)^2} e^{j\pi \alpha t^2} & \quad \sqrt{2} e^{-2\pi[(t/\sigma)^2 + (\sigma^2(f - \alpha t)^2]} \\
  \frac{1}{\sqrt{\sigma}} e^{-\pi((t-t_i)/\sigma)^2} e^{j2\pi f_i t} & \quad \sqrt{2} e^{-2\pi[((t-t_i)/\sigma)^2 + \sigma^2(f - f_i)^2]} \\
  \text{rect}_a(t) & \quad \frac{\sin[4\pi(a-|t|)f]}{\pi f} \text{rect}_a(t)
\end{align*}
\]

**Homework:** Proofs [3] 12.3.2 & [2] 10.4.2.2
WD: properties

\[
\begin{align*}
    x(t) & \quad W_x(f, t) \\
\frac{\sin(2\pi at)}{\pi t} & \quad \frac{\sin[4\pi(a - |f|)t]}{\pi t} \text{rect}_a(f) \\
e^{j \pi \alpha t^2} \text{rect}_a(t) & \quad \frac{\sin[4\pi(a - |f|)(f - \alpha t)]}{\pi (f - \alpha t)} \text{rect}_a(t) \\
\tilde{u}(t) = \begin{cases} 
1, & t > 0 \\
0, & t < 0 
\end{cases} & \quad \frac{\sin(4\pi ft)}{\pi f} \tilde{u}(t) \\
e^{-\sigma t} \tilde{u}(t) & \quad e^{-2\sigma t} \frac{\sin 4\pi ft}{\pi f} \tilde{u}(t) \\
u_n(t), \quad n = 0, 1, \ldots & \quad 2e^{-2\pi(t^2 + f^2)} L_n(4\pi(t^2 + f^2)) \\
\cos(2\pi f_i t) & \quad [\delta(f + f_i) + \delta(f - f_i) + 2\delta(f)\cos(4\pi f_i t)]/4 \\
\sin(2\pi f_i t) & \quad [\delta(f + f_i) + \delta(f - f_i) - 2\delta(f)\cos(4\pi f_i t)]/4 \\
\delta(t-t_i) + \delta(t-t_m) & \quad \delta(t-t_i) + \delta(t-t_m) + \\
& \quad 2\delta(t - \frac{t_i + t_m}{2}) \cos(2\pi(t_i - t_m)f) \\
e^{j2\pi f_i t} + e^{j2\pi f_m t} & \quad \delta(f - f_i) + \delta(f - f_m) + \\
& \quad 2\delta(f - \frac{f_i + f_m}{2}) \cos(2\pi(f_i - f_m)t)
\end{align*}
\]
The Ambiguity Function (AF)
The *autocorrelation* function

\[ r_x(\tau) = \int x(t + \tau)x^*(t)dt \]

can also be understood as the inverse Fourier transform of the energy density spectrum

\[ r_x(\tau) = \frac{1}{2\pi} \int X(\omega)X^*(\omega)e^{j\omega\tau} d\omega \]

[1]
AF: Autocorrelation

For the frequency-shifted signals we have

\[ \rho_x(\zeta) = \int x(t)x^*(t)e^{j\zeta t}dt \]
AF: Autocorrelation

For the frequency-shifted signals we have

$$\rho_x(\zeta) = \int x(t)x^*(t)e^{j\zeta t}dt$$

and

$$\rho_x(\zeta) = \frac{1}{2\pi} \int X(\omega)X^*(\omega + \zeta)d\omega$$

in the frequency domain.
AF: time and frequency shifted signals

Considering the signals

\[ x(t - \tau/2)e^{j(-\zeta/2)t} \]

and

\[ x(t + \tau/2)e^{j(\zeta/2)t} \]

as time and frequency shifted versions of one other, centred around \( x(t) \)...
The ambiguity function

We have the time-frequency autocorrelation function or *Ambiguity Function*

\[ A_x(\tau, \zeta) = \int x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{j\zeta t} dt \]
The ambiguity function

We have the time-frequency autocorrelation function or *Ambiguity Function*

\[
A_x(\tau, \zeta) = \int x \left( t + \frac{\tau}{2} \right) x^* \left( t - \frac{\tau}{2} \right) e^{j\zeta t} \, dt
\]

and in the frequency domain (via the Parseval’s relation)

\[
A_x(\tau, \zeta) = \frac{1}{2\pi} \int X \left( \omega - \frac{\zeta}{2} \right) X^* \left( \omega + \frac{\zeta}{2} \right) e^{j\omega\tau} \, d\omega
\]
The ambiguity function

We have the time-frequency autocorrelation function or *Ambiguity Function*

$$A_x(\tau, \zeta) = \int x \left( t + \frac{\tau}{2} \right) x^* \left( t - \frac{\tau}{2} \right) e^{j\zeta t} dt$$

and in the frequency domain (via the Parseval’s relation)

$$A_x(\tau, \zeta) = \frac{1}{2\pi} \int X \left( \omega - \frac{\zeta}{2} \right) X^* \left( \omega + \frac{\zeta}{2} \right) e^{j\omega \tau} d\omega$$

Interpretation as time-frequency autocorrelation...
Wigner distribution and ambiguity function

By setting $\zeta = 0$ we have

$$r_x(\tau) = A_x(\tau, 0)$$
Wigner distribution and ambiguity function

By setting $\zeta = 0$ we have

$$r_x(\tau) = A_x(\tau, 0)$$

and the energy density spectrum

$$S_x(\omega) = \int A_x(\tau, 0)e^{-j\omega \tau} d\tau$$
Wigner distribution and ambiguity function

For the autocorrelation function of the spectrum, with \( \tau = 0 \) we have

\[
\rho_x(\zeta) = A_x(0, \zeta)
\]
Wigner distribution and ambiguity function

For the autocorrelation function of the spectrum, with $\tau = 0$ we have

$$\rho_x(\zeta) = A_x(0, \zeta)$$

obtaining the temporal energy density

$$s_x(t) = \int A_x(0, \zeta)e^{-j\zeta t} d\zeta$$
Wigner distribution and ambiguity function

The transform with respect to \( \zeta \) yields the temporal autocorrelation function

\[
\phi_x(t, \tau) = \frac{1}{2\pi} \int A_x(\tau, \zeta) e^{-j \zeta t} d\zeta \\
= x \left( t + \frac{\tau}{2} \right) x^* \left( t - \frac{\tau}{2} \right)
\]
Wigner distribution and ambiguity function

The transform with respect to $\zeta$ yields the temporal autocorrelation function

$$\phi_x(t, \tau) = \frac{1}{2\pi} \int A_x(\tau, \zeta) e^{-j\zeta t} d\zeta$$

$$= x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2})$$

and thus the two-dimensional Fourier transform of $A_x(\tau, \zeta)$ is

$$W_x(t, \omega) = \frac{1}{2\pi} \int \int A_x(\tau, \zeta) e^{-j\omega \tau} e^{-j\zeta t} d\tau d\zeta$$
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\textit{Homework:} Signal reconstruction [1] 9.2.1

\textit{Homework:} Proof $W_x$ for $\delta(\cdot) \& e^{(\cdot)}$ [3] 12.3.2
WD: the Wigner-Ville distribution

In order to gain information on the stochastic process we define the Wigner-Ville spectrum as the expected value of the Wigner distribution:

\[ V_x(t,\omega) = \mathbb{E}[W_x(t,\omega)] = \int r_{xx}(t+\tau/2, t-\tau/2) e^{-j\omega\tau} d\tau \]
WD: the Wigner-Ville distribution

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\[ V_x(t, \omega) = \mathcal{E}[W_x(t, \omega)] = \int r_{xx} \left( t + \frac{\tau}{2}, t - \frac{\tau}{2} \right) e^{-j\omega\tau} d\tau \]

where

\[ r_{xx} \left( t + \frac{\tau}{2}, t - \frac{\tau}{2} \right) = \mathcal{E}[\phi_x(t, \tau)] = \mathcal{E} \left[ x \left( t + \frac{\tau}{2} \right) x^* \left( t - \frac{\tau}{2} \right) \right] \]
Altes Q distribution

The “wideband” version of the Wigner Distribution,
Altes Q distribution

The “wideband” version of the Wigner Distribution,

\[ Q_x(t, f) = f \int X( f e^{u/2} ) X^*( f e^{-u/2} ) e^{j2\pi t f u} du, \quad f > 0 \]
Altes Q distribution

The “wideband” version of the Wigner Distribution,

\[ Q_x(t, f) = \int X(f e^{u/2}) X^*(f e^{-u/2}) e^{j2\pi f u} du, \quad f > 0 \]

or the “scale-invariant” Wigner Distribution,

\[ Q_x(t, c) = \int x(t e^{\sigma/2}) x^*(t e^{-\sigma/2}) e^{-j2\pi c \sigma} d\sigma \]

by using the time-domain version of the signal \( x(t) \).
Altes Q distribution

Altes Q distribution was proposed for analyzing signals that have undergone compressions or dilations.
Altes Q distribution

Altes Q distribution was proposed for analyzing signals that have undergone compressions or dilations.

The Altes Q and the Wigner distributions are warped versions of each other

\[ Q_X(t,f) = W_{X^{++}} \left( \frac{tf}{fr}, fr \ln \frac{f}{fr} \right) \]
\[ W_X(t,f) = Q_{X^{--}} \left( te^{-f/fr}, fr e^{f/fr} \right) \]

where the signal is first prewarped for reference frequency \( fr \):
\[ X^{++}(f) = \sqrt{e^{f/fr}} X \left( fr e^{f/fr} \right) \text{ and } X^{--}(f) = \sqrt{\frac{fr}{f}} X \left( fr \ln \frac{f}{fr} \right), f > 0 \]
Cross-terms on quadratic TFR
Cross-terms on quadratic TFR

Consider the nonlinear operation

\[ |x(t) + y(t)|^2 = |x(t)|^2 + |y(t)|^2 + 2\Re \{x(t)y^*(t)\} \]
Cross-terms on quadratic TFR

Consider the nonlinear operation

\[ |x(t) + y(t)|^2 = |x(t)|^2 + |y(t)|^2 + 2\mathcal{R}\{x(t)y^*(t)\} \]

In a more general case, consider a multicomponent signal

\[ y(t) = \sum_{i=1}^{N} x_i(t) \]
Cross-terms on quadratic TFR

The WD of this signal is

\[ W_y(t, f) = \sum_{i=1}^{N} W_{x_i}(t, f) + 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} \Re \{ W_{x_i x_k}(t, f) \} \]

where

\[ W_{x_i x_k}(t, f) = \int x_i \left( t + \frac{T}{2} \right) x_k^* \left( t - \frac{T}{2} \right) e^{-j2\pi f \tau} d\tau \]
Cross-terms: example 1

For example, let be

\[ x_i(t) = x(t - t_i)e^{j2\pi f_it} \]
Cross-terms: example 1

For example, let be

\[ x_i(t) = x(t - t_i) e^{j2\pi f_i t} \]

using the properties of the WD we have

\[ W_y(t, f) = \sum_{i=1}^{N} W_x(t - t_i, f - f_i) + \ldots \]
Cross-terms: example 1

For example, let be

\[ x_i(t) = x(t - t_i)e^{j2\pi f_it} \]

using the properties of the WD we have

\[
W_y(t, f) = \sum_{i=1}^{N} W_x(t - t_i, f - f_i) \\
+ 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} W_x \left( t - \frac{t_i + t_k}{2}, f - \frac{f_i + f_k}{2} \right) \times \ldots
\]
Cross-terms: example 1

For example, let be

\[ x_i(t) = x(t - t_i)e^{j2\pi f_i t} \]

using the properties of the WD we have

\[ W_y(t, f) = \sum_{i=1}^{N} W_x(t - t_i, f - f_i) \]

\[ + 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} W_x \left( t - \frac{t_i + t_k}{2}, f - \frac{f_i + f_k}{2} \right) \]

\[ \times \cos \left( (f_i - f_k)t - (t_i - t_k)f + \frac{f_i + f_k}{2}(t_i - t_k) \right) \]
Cross-terms: example 1
Cross-terms: example 2
Outline

Introduction

Classical time-frequency representations
- Short-time Fourier transforms
- Wigner distribution
- Altes Q distribution
- Cross-term problems

Classes of TFRs
- Classes of TFRs with common properties
- Shift-covariant class
- Affine class
- Hyperbolic and kth power classes

Discrete calculations of TFRs
Classes of TFRs

To understand the relative advantages of TFRs it is useful to group them into *classes*.
Classes of TFRs

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Each class is defined by a few “ideal” TFR properties that all members must satisfy.
Classes of TFRs

To understand the relative advantages of TFRs it is usefull to group them into *classes*.

Each class is defined by a few “ideal” TFR properties that all members must satisfy.

Some of such classes may be:

- Shift-covariant or Cohen’s class
- Affine class
- Hyperbolic class
- Power class
## Classes of TFRs

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### Property

1. **Frequency Shift**
2. **Time Shift**
3. **Scale Covariance**
4. **Hyperbolic Time Shift**
5. **Convolution**
6. **Modulation**
7. **Real-Valued**
8. **Positivity**
9. **Time Marginal**
10. **Frequency Marginal**
11. **Energy Distribution**
12. **Time Moments**
13. **Frequency Moments**
14. **Finite Time Support**
15. **Finite Frequency Support**
16. **Instantaneous Frequency**
17. **Group Delay**
18. **Fourier Transform**
19. **Frequency Localization**
20. **Time Localization**
21. **Linear Chirp Localization**
22. **Hyperbolic Localization**
23. **Chirp Convolution**
24. **Chirp Multiplication**
25. **Moyal’s Formula**
### Classes of TFRs

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**Common Properties 2/2**
Shift-covariant (Cohen’s) class: definition
Shift-covariant (Cohen’s) class: definition

\[ C^c = \left\{ T_x(t, f) \mid y(t) = x(t - t_0)e^{j2\pi f_0 t} \right\} \]
Shift-covariant (Cohen’s) class: definition

\[ C^c = \left\{ T_x(t, f) \middle| y(t) = x(t - t_0)e^{j2\pi f_0 t} \Rightarrow T_y(t, f) = T_x(t - t_0, f - f_0) \right\} \]
Cc: alternative formulations

Any TFR within the Cohen’s class can be written in one of the following equivalent “Normal Forms”

\[ C^c_x(t, f; \Psi_c) \]
Cc: alternative formulations

Any TFR within the Cohen’s class can be written in one of the following equivalent “Normal Forms”

\[
C^c_x(t, f; \Psi_c) = \int \int \phi_c(t - \varsigma, \tau)x\left(\varsigma + \frac{\tau}{2}\right)x^*\left(\varsigma - \frac{\tau}{2}\right)e^{-j2\pi f\tau}d\varsigma d\tau
\]
Cc: alternative formulations

Any TFR within the Cohen’s class can be written in one of the following equivalent “Normal Forms”

\[
C^c_x(t, f; \Psi_c) = \int \int \phi_c(t - \varsigma, \tau) x \left( \varsigma + \frac{T}{2} \right) x^* \left( \varsigma - \frac{T}{2} \right) e^{-j2\pi f \tau} d\varsigma d\tau \\
= \int \int \Phi_c(f - \rho, \zeta) X \left( \rho + \frac{\zeta}{2} \right) X^* \left( \rho - \frac{\zeta}{2} \right) e^{-j2\pi t\zeta} d\rho d\zeta
\]
Cc: alternative formulations

Any TFR within the Cohen’s class can be written in one of the following equivalent “Normal Forms”

\[ C^c_{x}(t, f; \Psi_c) = \int \int \phi_c(t - \varsigma, \tau)x(\varsigma + \frac{\tau}{2})x^*(\varsigma - \frac{\tau}{2})e^{-j2\pi f \tau}d\varsigma d\tau \]
\[ = \int \int \Phi_c(f - \varrho, \zeta)X(\varrho + \frac{\zeta}{2})X^*(\varrho - \frac{\zeta}{2})e^{-j2\pi t \zeta}d\varrho d\zeta \]
\[ = \int \int \psi_c(t - \varsigma, f - \varrho)W_x(\varsigma, \varrho)d\varsigma d\varrho \]
Cc: alternative formulations

Any TFR within the Cohen’s class can be written in one of the following equivalent “Normal Forms”

\[
C^c_x(t, f; \Psi_c) = \int \int \phi_c(t - \varsigma, \tau) x \left( \varsigma + \frac{\tau}{2} \right) x^* \left( \varsigma - \frac{\tau}{2} \right) e^{-j2\pi f\tau} d\varsigma d\tau \\
= \int \int \Phi_c(f - \varrho, \zeta) X \left( \varrho + \frac{\zeta}{2} \right) X^* \left( \varrho - \frac{\zeta}{2} \right) e^{-j2\pi t\zeta} d\varrho d\zeta \\
= \int \int \psi_c(t - \varsigma, f - \varrho) W_x(\varsigma, \varrho) d\varsigma d\varrho \\
= \int \int \Psi_c(\tau, \zeta) A_x(\tau, \zeta) e^{-j2\pi (t\zeta - f\tau)} d\tau d\zeta
\]
Cc: alternative formulations

Any TFR within the Cohen’s class can be written in one of the following equivalent “Normal Forms”

\[
C_x^c(t, f; \Psi_c) = \int \int \phi_c(t - \varsigma, \tau)x\left(\varsigma + \frac{T}{2}\right)x^*\left(\varsigma - \frac{T}{2}\right)e^{-j2\pi f \tau}d\varsigma d\tau
\]

\[
= \int \int \Phi_c(f - \varrho, \zeta)X\left(\varrho + \frac{\zeta}{2}\right)X^*\left(\varrho - \frac{\zeta}{2}\right)e^{-j2\pi t \zeta}d\varrho d\zeta
\]

\[
= \int \int \psi_c(t - \varsigma, f - \varrho)W_x(\varsigma, \varrho)d\varsigma d\varrho
\]

\[
= \int \int \psi_c(\tau, \zeta)A_x(\tau, \zeta)e^{-j2\pi(t\zeta - f \tau)}d\tau d\zeta
\]

\[
= \int \int \gamma_c(f - f_1, f - f_2)X(f_1)X^*(f_2)e^{-j2\pi(f_1 - f_2)t}df_1df_2
\]

where the kernels are interrelated by...
Cc: alternative formulations

\[ \phi_c(t, \tau) = \int \int \Phi_c(f, \zeta) e^{-j2\pi(f\tau + t\zeta)} df d\zeta \]
Cc: alternative formulations

\[ \phi_c(t, \tau) = \int \int \Phi_c(f, \zeta) e^{-j2\pi(f\tau + t\zeta)} df d\zeta \]

\[ = \int \Psi_c(\tau, \zeta) e^{-j2\pi\zeta t} d\zeta \]
Cc: alternative formulations

\[ \phi_c(t, \tau) = \int \int \Phi_c(f, \zeta) e^{-j2\pi(f\tau + t\zeta)} df d\zeta \]

\[ = \int \Psi_c(\tau, \zeta) e^{-j2\pi \zeta t} d\zeta \]

\[ \xrightarrow{\mathcal{F}} \Phi_c(f, \zeta) \]
Cc: alternative formulations

\[ \psi_c(t, f) = \int \int \Psi_c(\tau, \zeta) e^{-j2\pi(t\zeta - f\tau)} df d\zeta \]
Cc: alternative formulations

\[ \psi_c(t, f) = \int \int \Psi_c(\tau, \zeta)e^{-j2\pi(t\zeta - f\tau)} df d\zeta \]
\[ = \int \Phi_c(f, \zeta)e^{-j2\pi \zeta t} d\zeta \]
Cc: alternative formulations

\[
\psi_c(t, f) = \int \int \Psi_c(\tau, \zeta) e^{-j2\pi (t\zeta - f\tau)} df d\zeta
\]

\[
= \int \Phi_c(f, \zeta) e^{-j2\pi \zeta t} d\zeta
\]

\[\xrightarrow{\mathcal{F}} \Psi_c(\tau, \zeta)\]
Cc: alternative formulations

\[
\psi_c(t, f) = \int \int \Psi_c(\tau, \zeta) e^{-j2\pi(t\zeta - f\tau)} df \, d\zeta
\]

\[
= \int \Phi_c(f, \zeta) e^{-j2\pi\zeta t} d\zeta
\]

\[
\Psi_c(\tau, \zeta) \xrightarrow{\mathcal{F}} \Phi_c(f, \zeta)
\]

and

\[
\Upsilon_c(f_1, f_2) = \Phi_c \left( \frac{f_1 + f_2}{2}, f_2 - f_1 \right)
\]
## TRFs into the Cohen’s class (1/3)

<table>
<thead>
<tr>
<th>Cohen’s-Class Distribution</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ackroyd</td>
<td>$ACK_x(t, f) = Re{x^*(t)X(f)e^{j2\pi ft}}$</td>
</tr>
<tr>
<td>Affine-Cohen Subclass</td>
<td>$AC_x(t, f; S_{AC}) = \int \int \frac{\sin(\pi \tau v)}{\pi \tau} A_F(\tau, v)e^{j2\pi (tv - f\tau)} d\tau dv$</td>
</tr>
<tr>
<td>Born-Jordon</td>
<td>$BJD_x(t, f) = \int \int \frac{\sin(\pi \tau v)}{\pi \tau} A_F(\tau, v)e^{j2\pi (tv - f\tau)} d\tau dv$</td>
</tr>
<tr>
<td>Butterworth</td>
<td>$BU D_x(t, f; M, N) = \int \int \left(1 + \left(\frac{\pi M}{\pi \tau}\right)^{2M} \left(\frac{\pi}{\pi \tau}\right)^{2N}\right)^{-1} A_F(\tau, v)e^{j2\pi (tv - f\tau)} d\tau dv$</td>
</tr>
<tr>
<td>Choi-Williams (Exponential)</td>
<td>$CWD_x(t, f; \sigma) = \int \int e^{-(2\pi \tau v)^2/\sigma} A_F(\tau, v)e^{j2\pi (tv - f\tau)} d\tau dv$</td>
</tr>
<tr>
<td>Cone Kernel</td>
<td>$CKD_x(t, f) = \int \int g(\tau)</td>
</tr>
<tr>
<td>Cumulative Attack Spectrum</td>
<td>$CAS_x(t, f) = \left</td>
</tr>
<tr>
<td>Cumulative Decay Spectrum</td>
<td>$CDS_x(t, f) = \left</td>
</tr>
<tr>
<td>Cohen’s-Class Distribution</td>
<td>Formula</td>
</tr>
<tr>
<td>----------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>Generalized Exponential</td>
<td>$G E D_x(t, f) = \int \int \exp \left[ -\left( \frac{\tau}{\tau_0} \right)^{2M} \left( \frac{v}{v_0} \right)^{2N} \right] AF_x(\tau, v) e^{j2\pi(tv-f\tau)} d\tau d\nu$</td>
</tr>
<tr>
<td>Generalized Rectangular</td>
<td>$G R D_x(t, f) = \int \int \text{rect}_1(</td>
</tr>
<tr>
<td>Generalized Wigner</td>
<td>$G W D_x(t, f; \vec{\alpha}) = \int x \left( t + \left( \frac{1}{2} + \vec{\alpha} \right) \tau \right) x^* \left( t - \left( \frac{1}{2} - \vec{\alpha} \right) \tau \right) e^{-j2\pi f\tau} d\tau$</td>
</tr>
<tr>
<td>Levin</td>
<td>$L D_x(t, f) = -\frac{d}{dt} \left</td>
</tr>
<tr>
<td>Margineau-Hill</td>
<td>$M H_x(t, f) = Re \left{ x(t)X^*(f) e^{-j2\pi ft} \right}$</td>
</tr>
<tr>
<td>Multiform</td>
<td>$M T_x(t, f; S) = \int \int S(\vec{\mu} \left( \frac{\tau}{\tau_0}, \frac{v}{v_0}; \alpha, r, \beta, \gamma \right)^{2k}) AF_x(\tau, v)e^{j2\pi(tv-f\tau)} d\tau d\nu$</td>
</tr>
<tr>
<td>Kernel</td>
<td>$S_{MTED}(\beta) = e^{-\pi\beta}, S_{MTBU}(\beta) = [1+\beta]^{-1}$</td>
</tr>
<tr>
<td>Nutall</td>
<td>$N D_x(t, f) = \int \int \exp \left{ -\pi \left[ \left( \frac{\tau}{\tau_0} \right)^2 + \left( \frac{v}{v_0} \right)^2 + 2r \left( \frac{\tau v}{\tau_0 v_0} \right) \right] \right} AF_x(\tau, v)e^{j2\pi(tv-f\tau)} d\tau d\nu$</td>
</tr>
<tr>
<td>Page</td>
<td>$P D_x(t, f) = 2 \ Re \left{ x^*(t)e^{j2\pi ft} \int_{-\infty}^{t} x(\tau)e^{-j2\pi f\tau} d\tau \right}$</td>
</tr>
<tr>
<td>Cohen’s-Class Distribution</td>
<td>Formula</td>
</tr>
<tr>
<td>---------------------------</td>
<td>---------</td>
</tr>
</tbody>
</table>
| **Pseudo Wigner**         | \[ PW_{D_x}(t, f; \Gamma) = \int x(t + \frac{\gamma}{2}) x^*(t - \frac{\gamma}{2}) e^{-j2\pi f \tau} d\tau \]  
  \[ = \int WD_x(0, f - f') WD_x(t, f') df' \] |
| **Reduced Interference**  | \[ RID_x(t, f) = \int \int \frac{1}{|t|} S\left(\frac{t-t'}{|t|}\right) x(t' + \frac{\tau}{2}) x^*\left(t' - \frac{\tau}{2}\right) e^{-j2\pi f \tau} dt' d\tau \]  
  with \( S(\beta) \in \mathbb{R}, \)  \( S(0) = 1, \)  \( \frac{d}{d\beta} S(\beta) \bigg|_{\beta=0} = 0, \)  \( S(\alpha) = 0 \) for \( |\alpha| > \frac{1}{2} \) |
| **Rihaczek**              | \[ RD_x(t, f) = x(t) X^*(f) e^{-j2\pi ft} \] |
| **Smoothed Pseudo Wigner**| \[ SPW_{D_x}(t, f; \Gamma, s) = \int s(t-t') PW_{D_x}(t', f; \Gamma) dt' \]  
  \[ = \int \int s(t-t') WD_x(0, f - f') WD_x(t', f') dt' df' \] |
| **Spectrogram**           | \[ SPEC_x(t, f; \Gamma) = \left| \int x(\tau) \gamma^*(\tau-t) e^{-j2\pi ft} d\tau \right|^2 = \left| \int X(f') \Gamma^*(f'-f) e^{j2\pi ft'} df' \right|^2 \] |
| **Wigner**                | \[ WD_x(t, f) = \int x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2}) e^{-j2\pi ft} d\tau = \int X(f + \frac{\tau'}{2}) X^*(f - \frac{\tau'}{2}) e^{j2\pi t f} df' \] |
Scale and time translation covariant or Affine class: definition
Scale and time translation covariant or Affine class: definition

\[ A^c = \left\{ T_x(t, f) \mid y(t) = \sqrt{|a|} x (a(t - t_0)) \right\} \]
Scale and time translation covariant or
Affine class: definition

\[ A^c = \left\{ T_x(t, f) \left| y(t) = \sqrt{|a|} x (a(t - t_0)) \right. \right\} \]

\[ \Rightarrow T_y(t, f) = T_x (a(t - t_0), \frac{f}{a}) \]
Ac: alternative formulations

$$A_c^x(t, f; \Psi_a)$$
Ac: alternative formulations

\[ A^c_x(t, f; \Psi_a) = |f| \int \int \phi_a(f(t - \varsigma), f\tau)x \left( \varsigma + \frac{T}{2} \right)x^* \left( \varsigma - \frac{T}{2} \right) e^{-j2\pi f\tau} d\varsigma d\tau \]
Ac: alternative formulations

\[
A^c_{x}(t, f; \Psi_a) = |f| \int \int \phi_a(f(t - \varsigma), f\tau)x(\varsigma + \frac{\tau}{2}) \ x^*(\varsigma - \frac{\tau}{2}) e^{-j2\pi f\tau} d\varsigma d\tau
\]

\[
= \frac{1}{|f|} \int \int \Phi_a \left(\frac{\varrho}{f}, \frac{\zeta}{f}\right) X \left(\varrho + \frac{\zeta}{2}\right) X^* \left(\varrho - \frac{\zeta}{2}\right) e^{-j2\pi t\zeta} d\varrho d\zeta
\]
Ac: alternative formulations

\[
A_c^x(t, f; \Psi_a) = |f| \int \int \phi_a(f(t - \varsigma), f\tau)x \left(\varsigma + \frac{T}{2}\right) x^* \left(\varsigma - \frac{T}{2}\right) e^{-j2\pi f\tau} d\varsigma d\tau
\]

\[
= \frac{1}{|f|} \int \int \Phi_a \left(\frac{\varrho}{f}, \frac{\zeta}{f}\right) X \left(\varrho + \frac{\zeta}{2}\right) X^* \left(\varrho - \frac{\zeta}{2}\right) e^{-j2\pi t\zeta} d\varrho d\zeta
\]

\[
= \int \int \psi_a \left(f(t - \varsigma), -\frac{\varrho}{f}\right) W_x(\varsigma, \varrho) d\varsigma d\varrho
\]
Ac: alternative formulations

\[ A_c^x(t, f; \Psi_a) = |f| \int \int \phi_a(f(t - \varsigma), f\tau)x \left( \varsigma + \frac{T}{2} \right) x^* \left( \varsigma - \frac{T}{2} \right) e^{-j2\pi f\tau} d\varsigma d\tau \]
\[ = \frac{1}{|f|} \int \int \Phi_a \left( \frac{\varrho}{f}, \frac{\zeta}{f} \right) X \left( \varrho + \frac{\zeta}{2} \right) X^* \left( \varrho - \frac{\zeta}{2} \right) e^{-j2\pi t\zeta} d\varrho d\zeta \]
\[ = \int \int \psi_a \left( f(t - \varsigma), -\frac{\varrho}{f} \right) W_x(\varsigma, \varrho) d\varsigma d\varrho \]
\[ = \int \int \Psi_a \left( f\tau, \frac{\zeta}{f} \right) A_x(\tau, \zeta) e^{-j2\pi(\varsigma f - \tau f)} d\tau d\zeta \]
Ac: alternative formulations

\[ A_c^x(t, f; \Psi_a) = |f| \int \int \phi_a(f(t - \varsigma), f\tau)x(\varsigma + \frac{T}{2})x^*(\varsigma - \frac{T}{2})e^{-j2\pi f\tau}d\varsigma d\tau = \frac{1}{|f|} \int \int \Phi_a \left( \frac{\varrho}{f}, \frac{\zeta}{f} \right)X \left( \varrho + \frac{\zeta}{2} \right)X^* \left( \varrho - \frac{\zeta}{2} \right)e^{-j2\pi t\zeta}d\varrho d\zeta \]

\[ = \int \int \psi_a \left( f(t - \varsigma), -\frac{\varrho}{f} \right) W_x(\varsigma, \varrho)d\varsigma d\varrho \]

\[ = \int \int \Psi_a \left( f\tau, \frac{\zeta}{f} \right) A_x(\tau, \zeta)e^{-j2\pi(t\zeta - f\tau)}d\tau d\zeta \]

\[ = \frac{1}{|f|} \int \int \gamma_a \left( \frac{f_1}{f}, \frac{f_2}{f} \right) X(f_1)X^*(f_2)e^{-j2\pi(f_1 - f_2)t}df_1 df_2 \]

where the kernels are interrelated by...
Ac: alternative formulations

\[ \phi_a(c, \nu) \overset{\mathcal{F}}{\leftrightarrow} \Phi_a(b, \beta) \]
Ac: alternative formulations

\[ \phi_a(c, \nu) \overset{\mathcal{F}}{\leftrightarrow} \Phi_a(b, \beta) \]

\[ \psi_a(c, b) \overset{\mathcal{F}}{\leftrightarrow} \Psi_a(\nu, \beta) \]
Ac: alternative formulations

\[ \phi_a(c, \nu) \leftrightarrow^\mathcal{F} \Phi_a(b, \beta) \]

\[ \psi_a(c, b) \leftrightarrow^\mathcal{F} \Psi_a(\nu, \beta) \]

and

\[ \Upsilon_a(b_1, b_2) = \Phi_a \left( -\frac{b_1 + b_2}{2}, b_1 - b_2 \right) \]
### TRFs into the Affine class (1/2)

<table>
<thead>
<tr>
<th>Affine Class Distrib.</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ackroyd</td>
<td>$ACK_x(t, f) = Re{x^*(t)X(f)e^{j2\pi ft}}$</td>
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<tr>
<td>Affine-Cohen Subclass</td>
<td>$AC_x(t, f; S_{AC}) = \int \frac{1}{</td>
</tr>
<tr>
<td>Affine-Hyp. Subclass</td>
<td>$AH_x(t, f; S_{AH}) = \int S_{AH}(f(t - t'))BP_0D_x(t'; f; \mu_0) dt'$</td>
</tr>
<tr>
<td>(Unitary) Bertrand $P_0$</td>
<td>$BP_0D_x(t, f; \mu_0) = \int X \left( f \frac{u/2}{\sinh(u/2)} \right) X^* \left( f \frac{u/2 - \mu/2}{\sinh(u/2)} \right) e^{j2\pi ft/u} du$</td>
</tr>
<tr>
<td>(General) Bertrand $P_0$</td>
<td>$BP_0D_x(t, f; \mu) =</td>
</tr>
<tr>
<td>Bertrand $P_x$</td>
<td>$BP_xD_x(t, f; \mu) = \int X(f \lambda_k(u))X^*(f \lambda_k(-u))\mu(u)e^{j2\pi tf(\lambda_k(u) - \lambda_k(-u))} du,$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_0(u) = \frac{u/2e^{u/2}}{\sinh(u/2)}, \lambda_1(u) = \exp\left[1 + \frac{ue^{u/2}}{e^{u/2} - 1}\right], \lambda_k(u) = \left[ \frac{e^{u/2k} - 1}{k-1} \right], k \neq 0, 1$</td>
</tr>
<tr>
<td>Born-Jordon</td>
<td>$BJD_x(t, f) = \int \frac{\sin(\pi tv)}{\pi tv} AF_x(t, v)e^{j2\pi (tv - f \tau)} d\tau dv$</td>
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<tr>
<td>Choi-Williams Exp.</td>
<td>$CWD_x(t, f; \sigma) = \int e^{-(2\pi tv)^2/\sigma} AF_x(t, v)e^{j2\pi (tv - f \tau)} d\tau dv$</td>
</tr>
<tr>
<td>Flandrin $D$</td>
<td>$FD_x(t, f) = \int X \left( f \left[ 1 + \frac{u^2}{4} \right] \right) X^* \left( f \left[ 1 - \frac{u^2}{4} \right] \right) \left[ 1 - \left( \frac{u^2}{4} \right) \right] e^{j2\pi ft/u} du$</td>
</tr>
</tbody>
</table>
TRFs into the Affine class (2/2)

<table>
<thead>
<tr>
<th>Affine Class Distr.</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Wigner</td>
<td>( GW_{D_{x}}(t, f; \tilde{\alpha}) = \int x \left( t + \left( \frac{1}{2} + \tilde{\alpha} \right) \tau \right) x^* \left( t - \left( \frac{1}{2} - \tilde{\alpha} \right) \tau \right) e^{-j2\pi ft} d\tau )</td>
</tr>
<tr>
<td>Localized Affine</td>
<td>( LA_{X}(t, f; G^{(A)}, F^{(A)}) = f \int X(f(-F^{(A)}(\beta) + \beta/2))X^*(f(-F^{(A)}(\beta) - \beta/2))G^{(A)}(\beta)e^{j2\pi ft} d\beta )</td>
</tr>
<tr>
<td>Margineau-Hill</td>
<td>( MH_{x}(t, f) = Re \left{ x(t)X^*(f) e^{-j2\pi ft} \right} )</td>
</tr>
<tr>
<td>Reduced Interference</td>
<td>( RID_{x}(t, f) = \int \int \frac{1}{</td>
</tr>
<tr>
<td></td>
<td>with ( S(\beta) \in \mathbb{N}, \ S(0) = 1, \frac{d}{d\beta}S(\beta)</td>
</tr>
<tr>
<td>Rihaczek</td>
<td>( RD_{x}(t, f) = x(t)X^*(f)e^{-j2\pi ft} )</td>
</tr>
<tr>
<td>Scalogram</td>
<td>( SCAL_{x}(t, f; \Gamma) = \left</td>
</tr>
<tr>
<td>Unterberger Active</td>
<td>( UAD_{X}(t, f) = f \int_{0}^{\infty} X(fu)X^*(fu)[1+u^{-2}]e^{j2\pi ft(u-1/u)} du )</td>
</tr>
<tr>
<td></td>
<td>( = f \int X \left( f \left( \sqrt{1+(\beta/2)^2} + \beta/2 \right) \right) X^* \left( f \left( \sqrt{1+(\beta/2)^2} - \beta/2 \right) \right) e^{j2\pi ft} d\beta )</td>
</tr>
<tr>
<td>Unterberger Passive</td>
<td>( UPD_{X}(t, f) = 2f \int_{0}^{\infty} X(fu)X^*(fu) \left[ \frac{1}{u} \right] e^{j2\pi ft(u-1/u)} du )</td>
</tr>
<tr>
<td></td>
<td>( = f \int X \left( f \left( \sqrt{1+(\beta/2)^2} + \beta/2 \right) \right) X^* \left( f \left( \sqrt{1+(\beta/2)^2} - \beta/2 \right) \right) \frac{1}{\sqrt{1+(\beta/2)^2}} e^{j2\pi ft} d\beta )</td>
</tr>
<tr>
<td>Wigner</td>
<td>( WD_{x}(t, f) = \int x \left( t + \frac{\tau}{2} \right) x^* \left( t - \frac{\tau}{2} \right) e^{-j2\pi ft} d\tau = \int X \left( f + \frac{v}{2} \right) X^* \left( f - \frac{v}{2} \right) e^{j2\pi tv} dv )</td>
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</table>
**Affine-Cohen sub-class: definition**

\[ AC_c(t,f) = \left\{ T_x(t,f) \mid y(t) = |a| x(a(t - t_0)) e^{j2\pi f_0 t} \Rightarrow T_y(t,f) = T_x(a(t - t_0),f - f_0 a) \right\} \]

**Homework**: Choi-William and Rihaczek distributions [4] 4.5.3
Affine-Cohen sub-class: definition

\[ AC^c = \left\{ T_x(t, f) \mid y(t) = \sqrt{|a|} x (a(t - t_0)) e^{j2\pi f_0 t} \right\} \]
Affine-Cohen sub-class: definition

\[
AC^c = \left\{ T_x(t, f) \middle| y(t) = \sqrt{|a|}x (a(t - t_0)) e^{j2\pi f_0 t} \right. \\
\Rightarrow T_y(t, f) = T_x (a(t - t_0), \frac{f-f_0}{a}) \right\}
\]
Affine-Cohen sub-class: definition

\[ AC^c = \left\{ T_x(t, f) \mid y(t) = \sqrt{|a|} x (a(t - t_0)) e^{j2\pi f_0 t} \right\} \]

\[ \Rightarrow T_y(t, f) = T_x (a(t - t_0), \frac{f-f_0}{a}) \]
TRFs into the Affine-Cohen sub-class

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</thead>
<tbody>
<tr>
<td>Generalized Wigner</td>
<td>[ GD_{x}(t, f; \tilde{\alpha}) = \int x \left( 1 + \left( \frac{1 + \tilde{\alpha}}{2} \right) \tau \right) x^* \left( 1 - \left( \frac{1 - \tilde{\alpha}}{2} \right) \tau \right) e^{-j2\pi f \tau} , d\tau ]</td>
</tr>
<tr>
<td>Localized Affine</td>
<td>[ LA_{x}(t, f; G^{(A)}, \beta) = f \int X \left( f \left( -F^{(A)}(\beta) + \frac{\beta}{2} \right) \right) X^* \left( f \left( -F^{(A)}(\beta) - \frac{\beta}{2} \right) \right) G^{(A)}(\beta) e^{j2\pi f \beta} , d\beta ]</td>
</tr>
<tr>
<td>Margineau-Hill</td>
<td>[ MH_{x}(t, f) = Re \left{ x(t) X^*(f) e^{-j2\pi f t} \right} ]</td>
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<td>Reduced Interference</td>
<td>[ RD_{x}(t, f) = \int \int \frac{1}{</td>
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<tr>
<td>with ( S(\beta) ) ( \equiv ) 1, ( S(0) = 1, \frac{d}{d\beta} S(\beta) \bigg</td>
<td>_{\beta = 0} = 0 ), ( \sigma(\alpha) = 0 ) for (</td>
</tr>
<tr>
<td>Rihaczek</td>
<td>[ RD_{x}(t, f) = x(t) X^*(f) e^{-j2\pi f t} ]</td>
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<tr>
<td>Scalogram</td>
<td>[ SCAL_{\alpha}(t, f; \Gamma) = \left</td>
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</tr>
<tr>
<td></td>
<td>[ = f \int X \left( f \left( \sqrt{1 + (\beta/2)^2} + \beta/2 \right) \right) X^* \left( f \left( \sqrt{1 + (\beta/2)^2} - \beta/2 \right) \right) e^{j2\pi f \beta} , d\beta ]</td>
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</tr>
<tr>
<td></td>
<td>[ = f \int X \left( f \left( \sqrt{1 + \left( \frac{\beta}{2} \right)^2} + \frac{\beta}{2} \right) \right) X^* \left( f \left( \sqrt{1 + \left( \frac{\beta}{2} \right)^2} - \frac{\beta}{2} \right) \right) \frac{1}{\sqrt{1 + (\beta/2)^2}} e^{j2\pi f \beta} , d\beta ]</td>
</tr>
<tr>
<td>Wigner</td>
<td>[ WD_{x}(t, f) = \int x\left( t + \frac{\tau}{2} \right) x^* \left( t - \frac{\tau}{2} \right) e^{j2\pi f \tau} , d\tau = \int X(f + \frac{\nu}{2}) X^* \left( f - \frac{\nu}{2} \right) e^{j2\pi f \nu} , d\nu ]</td>
</tr>
</tbody>
</table>
Hyperbolic class: definition


Hyperbolic class: definition

\[ H^c = \left\{ T_x(t, f) \right\} \]

\[ Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) \frac{1}{\sqrt{f}} e^{j2\pi c \ln(f/f_r)} \]
Hyperbolic class: definition

\[ H^c = \left\{ T_x(t, f) \left| Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) \frac{1}{\sqrt{f}} e^{j2\pi c \ln(f/f_r)} \right. \right\} \]

\[ \Rightarrow T_y(t, f) = T_x \left( a \left( t - \frac{c}{f} \right), \frac{f}{a} \right) \right\} \]

Remarks...
- Hyperbolic scale and time shift
Hyperbolic class: definition

\[ H^c = \left\{ T_x(t, f) \left| Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) \frac{1}{\sqrt{f}} e^{j2\pi c \ln(f/f_r)} \right. \right\} \]

\[ \Rightarrow T_y(t, f) = T_x \left( a \left( t - \frac{c}{f} \right), \frac{f}{a} \right) \right\} \]

Remarks...

- Hyperbolic scale and time shift
- Well suited for the analysis of self-similar random processes or wideband Doppler co-variant signals similar to the biosonar signals
Hyperbolic class: definition

\[ H^c = \left\{ T_x(t, f) \mid Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) \frac{1}{\sqrt{f}} e^{j2\pi c \ln(f/f_r)} \right\} \]

\[ \Rightarrow T_y(t, f) = T_x \left( a \left( t - \frac{c}{f} \right), \frac{f}{a} \right) \]

Remarks...
- Hyperbolic scale and time shift
- Well suited for the analysis of self-similar random processes or wideband Doppler co-variant signals similar to the biosonar signals
- Used by bats and dolphins for echolocation
Affine-hyperbolic sub-class: definition
Affine-hyperbolic sub-class: definition

\[ AH^c = \left\{ T_x(t, f) \right\} \quad Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) e^{j2\pi c \ln(f/f_r)} e^{j2\pi ft_0} \]
Affine-hyperbolic sub-class: definition

\[ AH^c = \left\{ T_x(t, f) \right\} Y(f) = \frac{1}{|a|} X \left( \frac{f}{a} \right) e^{j2\pi c \ln(f/f_r)} e^{j2\pi f t_0} \]

\[ \Rightarrow T_y(t, f) = T_x \left( a \left( t - t_0 - \frac{c}{f} \right), \frac{f}{a} \right) \]
κth power class: definition
$\kappa$th power class: definition

$$P_{\kappa}^c = \left\{ T_x(t, f) \left| Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) e^{\text{sgn}(f)j2\pi c|f/f_r|^\kappa} \right. \right\}$$
κth power class: definition

\[ P_{\kappa}^c = \left\{ T_x(t, f) \left| Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) e^{\text{sgn}(f)j2\pi c|(f/f_r)|^\kappa} \right. \right\} \]

\Rightarrow T_y(t, f) = T_x \left( a \left( t - \frac{\kappa}{f_r} \left| \frac{f}{f_r} \right|^{\kappa-1} \right), \frac{f}{a} \right) \]

Remarks...

- scale and power-time shift covariant
κth power class: definition

\[ P_{\kappa}^c = \left\{ \left. T_x(t, f) \right| Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) e^{\text{sgn}(f) j2\pi c |(f/f_r)|^\kappa} \right\} \]

\[ \Rightarrow T_y(t, f) = T_x \left( a \left( t - \frac{\kappa}{f_r} \left| \frac{f}{f_r} \right|^{\kappa-1} \right), \frac{f}{a} \right) \]

Remarks...

- scale and power-time shift covariant
- well matched to power chirps and dispersive power time shifts
\( \kappa \text{th power class: definition} \)

\[
P_{\kappa} = \left\{ T_x(t, f) \left| Y(f) = \frac{1}{\sqrt{|a|}} X \left( \frac{f}{a} \right) e^{\text{sgn}(f) j2\pi c |(f/f_r)|^\kappa} \right. \right\}
\]

\[
\Rightarrow T_y(t, f) = T_x \left( a \left( t - \frac{\kappa}{f_r} \left| \frac{f}{f_r} \right|^{\kappa-1} \right), \frac{f}{a} \right)
\]

Remarks...

- scale and \textit{power}-time shift covariant
- well matched to power chirps and dispersive power time shifts
- covariant to scale changes and the dispersive time shifts (that are proportional to powers of frequency)
Classes and its Common Properties
Outline

Introduction

Classical time-frequency representations
  Short-time Fourier transforms
  Wigner distribution
  Altes Q distribution
  Cross-term problems

Classes of TFRs
  Classes of TFRs with common properties
  Shift-covariant class
  Affine class
  Hyperbolic and kth power classes

Discrete calculations of TFRs
Discrete calculations of TFRs
Discrete calculations of TFRs

Sampling the signal...
Discrete calculations of TFRs

Sampling the signal...

...sampling the kernel...
Discrete calculations of TFRs

Sampling the signal...

...sampling the kernel...

... and defining discrete-time
and discrete-frequency distributions.
Discrete-time Wigner distribution

\[ W_x(n, e^{j\omega}) = 2 \sum_{m} x(n + m)x^*(n - m)e^{-j2\omega m} \]
Discrete-time Wigner distribution

We know that for discrete signal $x(n)$

$$X(e^{j\omega}) = X(e^{j\omega + k2\pi}), \ k \in \mathbb{Z}$$
Discrete-time Wigner distribution

We know that for discrete signal $x(n)$

$$X(e^{j\omega}) = X(e^{j\omega + k2\pi}), k \in \mathbb{Z}$$

However for the discrete-time WD we have the property

$$W_x(n, e^{j\omega}) = W_x(n, e^{j\omega + k\pi})$$
Discrete-time Wigner distribution

\[ W_x(n, e^{j\omega}) = 2 \sum_m x(n + m)x^*(n - m)e^{-j2\omega m} \]

Now, to avoid the aliasing in the discrete-time WD we need

\[ f_s \geq 4f_{max} \]
**Discrete-time Wigner distribution**

\[ W_x(n, e^{j\omega}) = 2 \sum_{m} x(n + m)x^*(n - m)e^{-j2\omega m} \]

Now, to avoid the aliasing in the *discrete-time* WD we need

\[ f_s \geq 4f_{max} \]

*Homework:* Fourier interpolation [4] 4.5.4
General discrete-time TFRs

The general discrete-time time-frequency distribution of Cohen's class is defined as

\[ C_x(n,k) = \sum_{m=-M}^{M} \sum_{\ell=-N}^{N} \phi(\ell,m)x(\ell+n+m)x^*(\ell+n-m)e^{-j4\pi km/L} \]

...considering only the discrete frequencies up to \( 2\pi k/L \), with \( L \) the DFT length.
General discrete-time TFRs

The general discrete-time time-frequency distribution of Cohen’s class is defined as

\[ C_x(n, k) = \sum_{m=-M}^{M} \sum_{\ell=-N}^{N} \phi(\ell, m)x(\ell+n+m)x^*(\ell+n-m)e^{-j\frac{4\pi km}{L}} \]

...considering only the discrete frequencies up to 2\(\pi k/L\), with \(L\) the DFT length.
The discrete-time Choi-Williams TFR

\[
\phi_{\text{CW}}(n,m) = \begin{cases} 
1 & \text{if } m = 0 \\
\delta(n) & \text{if } m \neq 0 
\end{cases}
\]

with

\[
\alpha_m = N \sum_{k=-N}^{N-1} \left| k \right| e^{-\sigma n^2/4m^2},
\]

and \(\sigma\) a free parameter.
The discrete-time Choi-Williams TFR

\[ \phi_{CW}(n, m) = \begin{cases} \frac{1}{|m| \alpha_m} e^{-\sigma n^2/4m^2}, & m \neq 0 \\ \delta(n), & m = 0 \end{cases} \]

with

\[ \alpha_m = \sum_{k=-N}^{N} \frac{1}{|m|} e^{-\sigma n^2/4m^2} \]

and \( \sigma \) a free parameter.
The discrete-time Choi-Williams TFR

With the normalization \( \sum_n \phi(n, m) = 1 \) the following interesting properties are preserved

\[
\sum_n CW_x(n, k) = |X(k)|^2
\]

and

\[
\sum_k CW_x(n, k) = |x(n)|^2
\]
Bibliography


